RESERVE AND EXCHANGE RATE CYCLES

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Abstract

Many Latin American countries appear locked into cycles of reserve loss, devaluation and temporary reserve gains. This paper shows how a dual exchange rate system with leakages may generate cycles in reserves and the premium between official and parallel exchange rates. We study the dynamics of these cycles and their asymptotic behavior both analytically and numerically.

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1. Introduction

In recent years, many Latin American countries have followed stop-go monetary and exchange rate policies. Often, reserve losses and balance of payments crises have provoked devaluations of the official exchange rate followed by periods in which reserves have been replenished. In many cases, a bewildering array of apparently quite different exchange rate institutions have been tried, combining fixed or crawling peg official exchange rates for trade-related transactions with black market or officially-sanctioned floating rate arrangements for capital account transactions.¹

How should one think of the sequence of regime switches and the cycles in foreign reserves and exchange rates that these countries have experienced? The standard approach has been to regard each failed stabilisation and subsequent devaluation as a separate episode. Studies such as Blanco and Garber (1986), Cumby and Van Wijnbergen (1989), Agenor and Delbecque (1991) and Agenor and Flood (1992) model the collapse of various exchange rate pegging arrangements in monetary models with rational expectations. Given the historical experience, however, it seems more reasonable to assume that markets expect a sequence of failed stabilisations stretching into the future.

Furthermore, the wide variety of exchange rate regimes that individual countries adopted at different times may be more apparent than real. In Argentina in the 1980s, for example, the succession of different exchange rate arrangements obscures the fact that for most of the period the authorities operated a de facto dual exchange rate market. This comprised (i) a centrally-organized market for exchange rate transactions associated with current-account-related payments, and (ii) a parallel market for capital-account-related payments which at some times was officially sanctioned and at others represented a black market in foreign exchange.²

¹Kamin (1991), Marion (1991), Agenor, Bhandari and Flood (1992) and Flood and Marion (1995) describe the experiences of a number of Latin American countries. See Agenor and Montiel (1996), chapter 8, for descriptions and a categorisation of different Latin American stabilisation policies. Also, see Edwards (1989).

²In this paper, we stress the use of dual exchange rate regimes in Latin America. In fact, they
Guided by these considerations, in this paper, we develop a theoretical model of reserve and devaluation cycles. Our approach extends past work by Rodriguez (1978), Dornbusch et al. (1983) and Kamin (1993). These authors discuss in an informal way steady-state cycles in dual exchange rate regimes. Ours is the first paper to analyze these cycles formally, solving for transition paths and studying their convergence properties. The starting point for our analysis is the assumption that the government adopts a fixed official exchange rate for trade-related transactions while there exists a parallel or dual foreign exchange market in which the exchange rate is freely determined. The parallel market may either represent an illegal black market or a formal, government-sanctioned market for capital account transactions.

The problem the authorities face in operating either formal or informal dual exchange rate systems is that of preventing ‘leakages’ between the two markets. Divergences between the rates prevailing in the two markets creates large incentives for agents to arbitrage by misrepresenting capital account transactions as payments associated with trade or vice versa. Bhandhari and Vegh (1990) amongst others have studied the ways in which, by systematically over and under-invoicing current account transactions, agents can disguise capital account transactions as trade-related and hence perform such arbitrages. In these circumstances, even if the true current account deficit is zero, the authorities will experience significant gains and losses in official reserves (respectively when the official rate is weaker or stronger than parallel market rate).

Agenor and Delbecque (1991) and Agenor, Bhandari and Flood (1992) show how reserve losses from leakages in a dual exchange rate model may lead to the eventual collapse of a fixed exchange rate peg. In the present paper, we show that such leakages have been widely applied elsewhere. After the collapse of Bretton-Woods in the early 1970s, several European countries, notably France, Italy and Belgium, introduced formal, dual exchange rates in an attempt to protect their reserves from speculative attacks while maintaining stable terms of trade. Many other countries in Asia and Africa have employed similar exchange arrangements in the last three decades. Marion (1991) describes European and Latin American experiences in detail and lists African and Asian countries that have employed multiple exchange rate systems. Pinto (1990) describes the use of dual exchange rates in sub-saharan Africa.
may, in fact, generate rich exchange and reserve dynamics in the form of cycles. We analyze both the ‘transition path’ or short-run dynamics of the resulting system and the steady-state cycles to which it tends. We look at comparative static properties of the regular cycles and study the government’s optimal choice of devaluation size. It turns out that the long run behavior of the system is path dependent in that it may settle down to quite different asymptotic cycles even for arbitrarily close initial conditions.

The paper is organized as follows. Section 2 provides motivation for the theory we subsequently develop by examining data on exchange rates, reserves and monetary aggregates of twelve Latin American countries. Our analysis suggests that exchange rate premia and foreign reserves levels presents cyclical co-movements consistent with the model presented in the paper. In Section 3, we present the model and supply regularity conditions for non-explosive dynamics. Section 4 characterizes regular cycles and provides convergence results. Section 5 analyses the government’s optimal choice of devaluation size. The conclusion summarizes the results of the paper and the appendix provides details of estimation procedures employed and proofs of lemmas and propositions.

2. Reserve cycles in Latin America

2.1 Data

In this section, we examine monetary and exchange rate data on twelve Latin American countries over the period 1970 to 1989 and deduce some stylized facts. The countries we study are Argentina, Bolivia, Chile, Costa Rica, Dominican Republic, Mexico, Paraguay, Peru, Uruguay, Brasil, Equador, and El Salvador. The data is quarterly, stretching from QI 1970 to QII 1989, and includes figures on official reserves, money and exchange rates. Data on reserves and monetary aggregates is expressed in US dollars and home currency respectively and comes from the IMF’s International Financial Statistics (IFS) database. The relevant codes for reserves and money are .1L.DZF... and 34...ZF... .
The exchange rate data is that employed by Marion (1994) and comes from the IFS and from International Currency Analysis (various issues). Throughout, rates are expressed as the US dollar price of one unit of the home currency. A crucial variable for our analysis is the exchange rate premium, defined as the difference between the parallel exchange rate (which in different countries at different times is either a legally sanctioned parallel rate or a black market rate) and the official exchange rate.

We start our analysis of this dataset by looking at correlation coefficients between money supply, reserves and exchange rate premia to establish some stylized facts. We shall see below that these are consistent with the implications of our theoretical model. Subsequently, we analyze the correlations in the frequency domain to see if there are indications of cycles.

2.2 Correlation coefficients

Table 1 shows correlations between the exchange rate premium (i.e., the parallel market rate minus the official rate) and changes in log reserves, denoted $\phi_{RP}$, for the twelve countries in our sample. The results suggest negative correlation between reserve changes and the exchange rate premium ($\rho_{RP}$ is negative in ten of the twelve countries studied). This finding is evidence in favour of the reserve leakages studied by Bhandari and Vegh (1990). When the parallel exchange rate is relatively high, the official rate is over-valued and agents can arbitrage by buying foreign exchange in the official market and selling it in the parallel market.

Table 1 also reports test statistics for the null hypothesis $\rho_{RP} = 0$ versus the alternative $\rho_{RP} \neq 0$. Under the null, the test statistics are distributed as Student’s t and hence values exceeding 2 allow one to reject the hypothesis of zero correlation at a 5% level. As one may see, individual t-statistics exceed 2 in relatively few cases. It is noticeable that the significance level of the tests is not reduced by splitting the

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$^3$To construct statistical tests, we use the fact that if $\phi$ and $\hat{\phi}$ are respectively the population and sample correlation coefficients between two variables, then $\zeta \equiv 1/2 \log((1 + \hat{\phi})/(1 - \hat{\phi}))$ is normal with mean $E(\zeta) = 1/2 \log((1 + \phi)/(1 - \phi))$, and variance $\text{Var}(\zeta) = 1/(T - 3)$. Moreover, if $\phi = 0$, then $|\hat{\phi}| \sqrt{(T - 2)/(1 - \hat{\phi}^2)}$ should be distributed as Student’s t with $T - 2$ degrees of freedom.
sample into sub-samples for the 1970s and 1980s.

Table 2 reports correlation coefficients, denoted $\phi_{PM}$, between the change in the log exchange rate premium (i.e., the difference between the official and the parallel exchange rates) and the change in log money. As one might expect, these correlations are consistently positive throughout the sample period and this result is statistically significant for the majority of countries. Table 3 gives correlation coefficients for changes in log reserves and changes in the log money stock, $\phi_{RM}$. Estimates of $\phi_{RM}$ based on the whole sample are consistently positive and significantly different from zero for four out of twelve countries. The positive correlation seems to disappear in the 1980s, however.

2.3 Spectral analysis

Since our topic is cycles in reserves and exchange rates, it is natural to consider the frequency domain equivalent of the correlation coefficients reported in the last section. The power spectrum,

$$s_x(\omega) \equiv \frac{1}{2\pi} \sum_{\tau=-\infty}^{+\infty} \text{cov}(x_t, x_{t-\tau}) \exp[-i\tau\omega] = \frac{C_0^x}{2\pi} + \frac{1}{\pi} \sum_{\tau=0}^{\infty} \text{cov}(x_t, x_{t-\tau}) \cos(\tau\omega), \quad (1)$$

of a stochastic process, $x_t$, may be thought of as a decomposition of its variance into contributions from different frequencies, $\omega$. The corresponding concept for the correlation coefficient between two processes, $x_t$ and $y_t$, discussed in the last section is their coherence, defined as

$$\text{Coh}(\omega) \equiv \frac{|s_{xy}(\omega)|^2}{s_x(\omega)s_y(\omega)}, \quad (2)$$

where the cross spectrum, $s_{xy}(\omega)$, is defined as

$$s_{xy}(\omega) \equiv \frac{1}{2\pi} \sum_{\tau=-\infty}^{+\infty} \text{cov}(x_t, y_{t-\tau}) \exp[-i\tau\omega]. \quad (3)$$

The fact that the spectrum is non-negative implies that the coherence is positive and one may show that, in fact, it lies in the interval $[0, 1]$. A high value of $\text{Coh}(\omega)$ indicates that the two processes have a strong positive relation at the frequency $\omega$. 
Given that we employ quarterly data, the length of cycle in years corresponding to a frequency of $\omega$ is $\pi/\omega/2$.

The approach we take to estimating coherences is described in the Appendix. It basically consists of replacing the covariances that appear in equations (1) and (3) with their sample equivalents and then multiplying the terms in the summations by weights that decrease as the lag length rises. The results of these estimations are given in Figures 1 and 2 which respectively show the coherences between reserves and domestic credit and reserves and the exchange premium for data from the 1970s and the 1980s.

The plots based on 1970s data exhibit very little systematic pattern in the coherences. Only for Chile is there much correspondence between the two coherences shown, with the tri-modal pattern for the reserves-money coherence also appearing in the reserves-exchange premium coherence. In contrast, the results for the 1980s suggest much closer correspondence between the two coherences. For most countries, the coherences track each other closely in this second period and multiple peaks are suggestive of cyclical behavior.

For most countries in the 1980s, there appears to be a low frequency cycle of about $\pi/0.6=5.2$ years duration. Many also exhibit higher frequency cycles. For example, the coherences for the Dominican Republic also peak at 0.8 and 1.6, suggesting cycle lengths of $\pi/1.6=2.0$ and $\pi/3.2=1.0$ years. For Bolivia, the coherence estimates indicate cycle lengths of $\pi/0.4=7.9$, $\pi/2=1.6$ and $\pi/5.6=0.6$ years.

2.4 Stylized facts

To summarize, the stylized facts suggested by our analysis of correlations between money, reserves and exchange rate premia are:

1. that reserves fall when the exchange rate premium is relatively large,
2. that the exchange premium is increasing with the real money supply,
3. that no clear pattern of correlation exists between reserves and money supply,
4. that reserves, money and exchange premia seem to cycle together in many Latin American countries in the 1980s, but that no similar pattern is apparent in data from the 1970s,

5. that most countries in our sample exhibit a low frequency and one or two higher frequency cycles, with typical cycle lengths being 5 years for the former and 0.8-1.5 years for the latter.

3. The model

3.1 Basic assumptions

In this section, we consider a theoretical model of dual exchange rates with leakages consistent with the findings of our empirical analysis of data from Latin American countries. For simplicity, we work with a standard monetary model of exchange rate determination. The single non-standard element is the assumption that residents are obliged to carry out current account transactions with non-residents at a fixed exchange rate, $c_t$, while other transactions are performed at a freely floating rate, $s_t$.

The existence of two rates creates incentives for agents to arbitrage. Such arbitrage generates reserve gains or losses even if the true current account deficit is zero. A reasonable approach to modeling money market equilibrium is, therefore, to assume that the rate of reserve loss is an increasing function of the premium between the two exchange rates. Incorporating such reserve dynamics in a standard, monetary model of exchange rate determination yields the system

\begin{align*}
\text{Money demand} & \quad m_t = \ p_t - \alpha \dot{s}_t, \\
\text{Money supply} & \quad m_t = \lambda (c_t + R_t) + (1 - \lambda) D_t, \\
\text{Purchasing power parity} & \quad p_t = \theta s_t + (1 - \theta) c_t, \\
\text{Reserve dynamics} & \quad \dot{R}_t = -\gamma (s_t - c_t), \\
\text{Domestic credit growth} & \quad \dot{D}_t = \mu,
\end{align*}

(4) (5) (6) (7) (8)

where $\alpha, \lambda, \theta, \gamma$, and $\mu$ are constants, satisfying $0 < \lambda < 1$, $\gamma > 0$, and $0 < \theta < 1$. 

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Variables are expressed in natural logarithms. $m_t$ is the nominal money stock, $p_t$ is the domestic price level, $D_t$, the stock of domestic credit, $c_t$ is the official exchange rate, $s_t$ is the parallel rate, and $R_t$ is the value of the central bank’s foreign exchange holdings expressed in foreign currency.

Equation (4) gives the demand for money balances as a function of the expected depreciation in the parallel exchange rate (the interest rate has been substituted out using uncovered interest parity with the foreign interest rate normalized to zero). Equation (5) gives the supply of money in the form of a log linear approximation. Normalizing the foreign price level to unity, equation (6) links domestic prices to the two exchange rates by way of a purchasing power condition. Equation (7) gives the dynamic equation for reserves. When the parallel is weaker than the official rate, the country loses reserves, and vice versa when the official rate is the weaker. Finally, equation (8) specifies the dynamics of domestic credit in the form of a smooth time trend. One may think of $D_t$ as the basic forcing variable in this model. Assuming a smooth trend abstracts from any successful attempts by the authorities to reduce domestic credit growth by fiscal or monetary stringency.

The crucial extra element that we need to specify is the dynamics of the quasi-fixed, official exchange rate, $c_t$. Suppose that $c_t$ is constant until $R_t = R$, at which point it jumps up by a fixed amount, $\delta$. Thus, we assume that devaluations in the fixed exchange rate are triggered by reserves hitting a minimum level. Below, we examine the government’s choice of an optimal devaluation size. In fact, the model we develop below may be extended with almost no changes to the case in which each

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4Note that this equation implies that the premium must become negative for reserves to increase. Examination of data reveals that for many countries, the premium was consistently positive during the 1980s. To capture this, the model can easily be modified by including a constant on the right hand side of equation (7) in addition to the term proportional to $(s_t - c_t)$. Such a constant might represent the effect on reserves of underlying trade and long-term capital balances.

5Our use of a purely monetary model, excluding any interaction between real and nominal quantities due, for example, to wage or price inertia, seems justified since we are interested in explaining developments in developing countries in which monetary growth and inflation often reach very high rates. Agenor and Flood (1992) consider a model of the collapse of a dual exchange rate system and conclude that, broadly speaking, results are the same with and without sticky prices.
devaluation magnitude is a positive, independent random variable with mean $\delta$.

3.2 Solving for the dynamic paths

To analyze the system, first redefine the state variables to ensure stationarity of the solutions. A natural way to rewrite the nominal quantities in the model is in terms of foreign goods prices, i.e. deflating by the official exchange rate. Thus, define $Z_t \equiv D_t - c_t$, and $\rho_t \equiv s_t - c_t$. The fact that the exchange rate is a forward-looking variable and the absence of any explicit dependence on time means that, at any given moment, the exchange rate premium is a function of the levels of $R_t$ and $Z_t$, i.e. $\rho_t \equiv \rho(R_t, Z_t)$. We shall find it convenient below to regard $\rho_t$ sometimes as a function of time and sometimes as a function, $\rho(R_t, Z_t)$, of the state variables.

Let us now consider the dynamics of the system given the above assumptions. Between devaluations of the official exchange rate, the system obeys the vector differential equation:

$$
\begin{bmatrix}
\dot{\rho}_t \\
\dot{R}_t
\end{bmatrix} =
\begin{bmatrix}
\theta/\alpha & -\lambda/\alpha \\
-\gamma & 0
\end{bmatrix}
\begin{bmatrix}
\rho_t \\
R_t
\end{bmatrix} +
\begin{bmatrix}
(\lambda - 1)/\alpha \\
0
\end{bmatrix}
Z_t \equiv \Phi_1 \begin{bmatrix}
\rho_t \\
R_t
\end{bmatrix} + \Phi_2 Z_t,
$$

where, subject to our assumptions and given $Z_0$, $Z_t$ is an exogenous forcing process with known path.

We shall begin by solving this system as though it were an initial value problem, i.e., for known initial conditions, $\rho_0$, $R_0$ and $Z_0$. Of course, this does not constitute a full solution as the initial premium, $\rho_0$, is actually a function of initial reserve and domestic credit levels, $R_0$ and $Z_0$. Below, we shall describe the boundary condition associated with the government’s devaluation policy (i.e., devalue by $\delta$ whenever $R_t = R$). This boundary condition will pin down the initial exchange rate, $\rho_0$ as a function of $R_0$ and $Z_0$, thereby yielding a function $\rho_t = \rho(R_t, Z_t)$.

Solving the differential equation system for given initial $\rho_0$ yields the following proposition.
Proposition 1. Suppose that $\rho_0$, $R_0$ and $Z_0$ are known initial values. The solution to the dynamic system given in equation (9) up to the date $T$ at which $R_t$ first equal $R$ is the sum of (i) the solution to the homogeneous part of the above differential equations, $(\tilde{\rho}_t, \tilde{R}_t)'$, plus (ii) a particular solution to the non-homogeneous equation, denoted $\rho^*_t$, and $R^*_t$. (i) equals:

\[
\begin{align*}
\tilde{\rho}_t &= A_1 \exp(\eta_1 t) \xi_{11} + A_2 \exp(\eta_2 t) \xi_{12}, \\
\tilde{R}_t &= A_1 \exp(\eta_1 t) \xi_{21} + A_2 \exp(\eta_2 t) \xi_{22},
\end{align*}
\]

where $\eta_1$, $\eta_2$ are the eigenvalues of $\Phi_1$, and $(\xi_{11}, \xi_{21})'$ and $(\xi_{12}, \xi_{22})'$ are the corresponding eigenvectors, and $A_i$, $i = 1, 2$ are undetermined constants. An example of (ii) is

\[
\begin{align*}
\rho^*_t &= C_1 \int_0^t \exp(\eta_1 (t - \tau)) \xi_{11} Z_\tau d\tau + C_2 \int_0^t \exp(\eta_2 (t - \tau)) \xi_{12} Z_\tau d\tau, \\
R^*_t &= C_1 \int_0^t \exp(\eta_1 (t - \tau)) \xi_{21} Z_\tau d\tau + C_2 \int_0^t \exp(\eta_2 (t - \tau)) \xi_{22} Z_\tau d\tau,
\end{align*}
\]

where the $C_i$, $i = 1, 2$ are chosen so that:

\[
\begin{align*}
C_1 \xi_{11} + C_2 \xi_{12} &= (\lambda - 1)/\alpha, \\
C_1 \xi_{21} + C_2 \xi_{22} &= 0.
\end{align*}
\]

$(A_1, A_2)'$ are then equal to

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} = \begin{bmatrix}
\xi_{11} & \xi_{12} \\
\xi_{21} & \xi_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
\rho_0 \\
R_0
\end{bmatrix}.
\]

To understand the form of the solutions, it helps to examine the plot given in the upper panel of Figure 3. The plot shows orbits of the system in $(R, Z)$-space. Since the exchange rate premium, $\rho_t$, is a function of $R_t$ and $Z_t$, one may fully describe the dynamics in terms of these two variables. Note that the systems shown in the Figure are solved for a devaluation reserve trigger level $R$ of 0.

Now, consider a typical orbit starting after a devaluation. Initially, reserves accumulate as we move along the new orbit (from left to right in the figure). The rise in
domestic credit, $Z_t$, leads to a progressive depreciation in the parallel exchange rate, $s_t$, and hence a rise in the exchange rate premium, $\rho_t$. Eventually, reserves cease to accumulate and indeed start falling. When $R_t$ reaches $\bar{R}$, a devaluation is triggered and $c_t$ jumps up by $\delta$. Since the devaluation is fully anticipated, and the absence of arbitrage precludes anticipated jumps in the parallel exchange rate, $s_t$, this implies that $\rho_t$ must decline by $\delta$. In turn, since $D_t$ is predetermined, domestic credit valued in foreign currency, $Z_t \equiv D_t - c_t$ must jump down by $\delta$.

These economic arguments, i.e., the absence of arbitrage and the way it affects the system when the official rate is devalued, supply a boundary condition that determines $\rho_0$. To restate this mathematically, consider the solution starting an instant after a devaluation. At such a moment, the particle lies on the line segment between $(Z^* - \delta, R)$ and $(Z^*, R)$, where $Z^*$ is defined implicitly by $\rho(R, Z^*) = 0$ and represents the highest level of domestic credit such that reserves are accumulating when $R = R$.

Suppose that we know $\rho(R, z)$ for all $\{z : (R, z) \in [(R, Z^* - \delta), (R, Z^*)]\}$, then, by running the dynamic system round until each of the orbits again hits the $R$ line, we can find the values of $Z_T$ and $\rho_T$ at those points. One can then see if the levels of $\rho$ immediately before a devaluation are consistent with the initial guess for $\rho$ on $[(R, Z^* - \delta), (R, Z^*)]$ by devaluing a sufficient number of times until the particle lies again in $[(R, Z^* - \delta), (R, Z^*)]$, and checking that $\rho_T - i\delta = \rho(R, Z_T - i\delta)$, where $i$ is the integer number of times we have to devalue to be on $[(R, Z^* - \delta), (R, Z^*)]$.

The above argument may be stated in the form of a proposition:

**Proposition 2.** A cyclical equilibrium for the model described in equations (4) to (8) consists of the solution given in Proposition 1 together with a function, $\rho_0(Z)$ defined for $Z \in [Z^* - \delta, Z^*]$ for a constant $Z^*$. $\rho_0$ should be thought of as the value of the exchange rate premium immediately following devaluations. $\rho_0$ has the property that, for any $Z_0 \in [Z^* - \delta, Z^*]$, if we define $(Z_1, \rho_1)$ as the values of $Z_t$ and $\rho_t$ on the

Clearly this two point boundary problem is defined properly if condition (16) holds anytime a devaluation takes place. This will certainly be the case if in (16) we replace $Z_0$ with $Z^*$.
first time that \( R_t = R \), then if \( i \) is the smallest integer such that \( Z_1 - i\delta < Z^* \), 
\[ \rho_1 - i\delta = \rho_0(Z_1 - i\delta). \]

The way in which we have exposited the above dynamic solution immediately suggests how one may go about solving numerically for \( \rho_0(Z) \). Guessing a function \( \rho_0(Z) \), and a constant \( Z^* \), we can calculate the dynamic path starting at \((R, Z, \rho_0(Z))\). ‘Running this out’ until the first time, \( T > 0 \), at which \( R_t \) hits the trigger reserve level, \( R \), we can check to see if \( \rho_T - i\delta = \rho_0(Z_T - i\delta) \). One may then iterate on the initial function, \( \rho_0(.) \), until a fixed point is reached. The main challenges in solving this numerically are (i) that one is searching for the fixed point in a class of functions, \( \rho_0(.) \), and (ii) that \( Z^* \) is not given and must be solved for simultaneously. The latter feature implies that we are dealing with a free boundary problem.

For a cyclical equilibrium to exist, the orbits of the dynamic system must curve round and return to \( R \) as \( Z \) increases.\(^7\) To ensure this, one must rule out explosive dynamic paths along which reserves are accumulated endlessly. A simple sufficient condition to exclude explosive behaviour is:

**Lemma 1.** Let \( \eta_1 \) be the positive eigenvalue of the dynamic system (9) and \( Z_0 \) the initial domestic credit level. Then, \( R_t \) will reach the trigger value \( R \) if the following condition holds:

\[
Z_0 + \frac{A_1 \eta_1}{C_1} + \frac{\mu}{\eta_1} < 0, \tag{16}
\]

where \( A_1 \) and \( C_1 \) are defined in Proposition 1.

Given that we suppose \( R_t \geq R \), noncyclical equilibria can only exist if the condition in Lemma 1 does not hold. In this case, if we impose a no bubbles condition, the equilibrium must be linear and will involve reserves growing without limit.

Finally, consider how the model described above changes if devaluation size is stochastic. Above, we assumed a deterministic devaluation size of \( i\delta \), where \( i \) is the

\(^7\)That is, starting at some initial point, \((R, Z_0)\), the dynamic path must eventually return to the \( R = R \) line at a higher level of \( Z \).
smallest integer such that the change in reserves is positive after the devaluation occurs. Now, let the devaluation size be randomly with a distribution which has positive support on \([Z^* - \delta, Z^*]\) and mean \(i\delta\) where \(i\) is defined as before. If agents are risk neutral, then the solutions of the model will be exactly the same as before. The only difference is that when devaluations take place, the particle will jump to some randomly determined point on \([Z^* - \delta, Z^*]\).

4. Long run cycles

4.1 Steady state cycles

We now turn to the long run, steady state behavior of the model described above. We first adopt the definition:

Definition 1. The system is in a steady state cycle of duration \(T < \infty\) if, for some \(t\), \((R_t, Z_t) = (R_{t+T}, Z_{t+T}) = (R, Z)\) and \((R_s, Z_s) \neq (R, Z)\) for all \(s \in (t, t + T)\). The cycle is of order \(N\) if there are \(N\) devaluations in \([t, t + \Delta]\).

The most obvious kind of steady state cycle is of order 1 and is depicted in Figure 4. These resemble the steady-state cycles studied by Rodriguez (1978), and Kamin (1993). In a slightly different context, Dornbusch et al. (1983) also studied this type of cycle.

It is obvious that the duration of an order 1 cycle, denoted \(T\), must equal \(T = \delta/\mu\). This is a simple consequence of the fact that, between devaluations, \(Z_t\) grows at the exogenous rate, \(\mu\), and hence if the last devaluation occurred at \(t = 0\), then \(Z_{T+} = Z_{0+} = Z_{0-} - \delta\). To solve completely for the order 1 cycle requires determination of \(\rho_0\), however. For this, one may use the fact that, \(\rho_T = \rho_0 + \delta\), where \(T\) is defined as above and \(t = 0\) is the instant after devaluation. Normalizing so that \(R = 0\) and using the solutions given in Proposition 1, one may solve for the \((Z_0, \rho_0)\), which pertain at the start of each cycle are.
Proposition 3. The levels of \( Z_t \) and \( \rho_t \) at the start of each order 1 steady state cycle, \((Z_0, \rho_0)\), are:

\[
\begin{bmatrix}
\rho_0 \\
Z_0
\end{bmatrix} = \Sigma^{-1} \begin{bmatrix}
\delta + \frac{C_1 \xi_{11}}{\eta_1} (1 - \exp(\eta_1 T)) + \frac{C_1 \xi_{12}}{\eta_2} (1 - \exp(\eta_2 T)) \\
C_1 \xi_{21} \frac{\mu T}{\eta_1} + \frac{C_1 \xi_{21}}{\eta_1} (1 - \exp(\eta_1 T)) + C_2 \xi_{22} \frac{\mu T}{\eta_2} + \frac{C_1 \xi_{22}}{\eta_2} (1 - \exp(\eta_2 T))
\end{bmatrix}
\]

where \( C_i, i = 1, 2 \) are as defined in Proposition 1 and

\[
\Sigma \equiv \begin{bmatrix}
\xi'_{11} \exp(\eta_1 T) \xi_{11} + \xi'_{21} \exp(\eta_2 T) \xi_{12} - 1 & \frac{C_1 \xi_{11}}{\eta_1} (\exp(\eta_1 T) - 1) + \frac{C_2 \xi_{12}}{\eta_2} (\exp(\eta_2 T) - 1) \\
\xi'_{11} \exp(\eta_1 T) \xi_{21} + \xi'_{21} \exp(\eta_2 T) \xi_{22} & \frac{C_1 \xi_{21}}{\eta_1} (\exp(\eta_1 T) - 1) + \frac{C_2 \xi_{22}}{\eta_2} (\exp(\eta_2 T) - 1)
\end{bmatrix}.
\]

Here, \( \xi'_{ij} \) is the \( i, j \)th element of the inverse of the matrix whose columns are equal to the eigenvectors of \( \Phi \) and \( \Sigma \) is supposed to be non-singular.

It is important to realize that steady state cycles other than those of order 1 are possible, however. If one examines the pattern of orbits in Figure 3, it is fairly obvious that for some \( \delta \), there will exist steady state cycles ‘of order 2’ in which the system follows a small followed by a large orbit, returning then to the start of the smaller orbit.\(^8\)

4.2 Convergence to steady state cycles

This section studies the convergence properties of dual exchange rate systems of the kind described above.\(^9\) Under the following assumption, we obtain simple and surprisingly strong results:

Assumption 1. All devaluations are of size \( \delta \).

\(^8\)Solving for the steady state order-2 cycle is rather more difficult and requires numerical methods. For the order-1 cycle, we can write the cycle duration, \( T \), explicitly in terms of exogenous parameters, \( \delta \) and \( \mu \), i.e., \( T = \delta / \mu \). There are no comparably simple expressions for the durations of the two orbits in the order-2 cycle.

\(^9\)Our analysis is based on qualitative features of the dynamic system and is hence applicable to broader class of cycling, dual exchange rate models, that the linear model described by equations (4) to (8).
Note this is restrictive compared with Proposition 2 in which we supposed that devaluations magnitudes were \( i\delta \), where \( i \) is an integer and \( i\delta \) is the smallest devaluation such that reserves will be increasing after devaluation has occurred.

**Proposition 4.** Under Assumption 1, if the economy converges to a steady state cycle of order \( N \), then \( N < 3 \). If an order 2 steady state cycle exists, then the economy exhibits path dependence, in that there are initial conditions, \((Z_{t_0}, R_{t_0}) = (Z_1, R)\), and \((Z_{t_0}, R_{t_0}) = (Z_2, R)\), arbitrarily close together, for which the system will converge to order 1 and order 2 cycles, respectively.

Here, we show that, the economy’s asymptotic behaviour must be one of two types. Either reserves and exchange rate premia cycle in a completely regular way, following exactly the same path repeatedly, or the economy will follow an order 2 cycle, with short and long inter-devaluation times occurring successively.

In fact, whether or not the economy is moving towards an order 1 or an order 2 steady state cycle can be judged from the sequence of inter-devaluation times even when the economy is not in steady state. Results obtained in the course of proving Proposition 4 (see the Appendix) imply the following Corollary.

**Corollary.** Suppose the economy is not in a steady state cycle and that Assumption 1 holds.

1. If the periods between devaluations tend to increase or decrease, in the sense that given three such successive periods \((T_1, T_2, T_3)\), the second and third are both either larger or smaller than the first (either \( T_1 < T_2, T_3 \) or \( T_1 > T_2, T_3 \)), then the economy is moving towards an order 1 cycle.

2. On the other hand, if periods between devaluations fluctuate in that for any three such periods, the magnitude of the first lies between those of the second and third (either \( T_2 < T_1 < T_3 \) or \( T_3 < T_1 < T_2 \)), then the economy is converging to an order 2 cycle.
4.3 Order 1 cycle comparative statics

To see the relative significance of different parameters, it is interesting to perform comparative statics. In Table 4, we report elasticities of the maxima and minima over the regular order 1 cycle of the log exchange rate premium, \( \rho \), log reserves, \( R_t \), and the log of domestic credit at foreign prices, \( Z_t \). The variable most sensitive to changes in parameters is \( Z_t \). Increasing the demand semi-elasticity of money, \( \alpha \), the money supply log linearization parameter, \( \lambda \), or the rate of domestic credit growth, \( \mu \), shifts up the minimum and maximum \( Z_t \) over the regular cycle quite substantially. As one might expect, raising the devaluation size increases and lowers the maximum and minimum levels of \( Z_t \) over the cycle.

The amplitude of reserve cycles, as measured by the difference between the maximum \( R_t \) and \( R_t^* \), is surprisingly insensitive to changes in parameters. The elasticity of reserve cycle amplitude to devaluation size is a mere 0.09, while other parameters have even smaller elasticities. Increasing \( R_t \) has no significant impact on reserve cycle amplitude since the maximum reserve level rises almost one for one. The minimum and maximum levels of \( \rho_t \) over the cycle are also fairly insensitive to parameters although changes in the log linearization parameters of money supply and of the purchasing power parity condition do have some impact.

5. Optimal devaluation policy

In the 1980s, most stabilization programmes in Latin America were frustrated by the authorities’ inability to control monetary aggregates and public spending.\(^{10}\) Within our model, we represent this by assuming an exogenously given rate of domestic credit growth, \( \mu \), outside the control of the central bank. On the other hand, we suppose that the monetary authorities can freely determine the amount they devalue, \( \delta \), when reserves hit their minimum level, \( R_t^* \). In this section, we study factors which affect the central bank’s optimal choice of \( \delta \).

\(^{10}\)See Agenor and Montiel (1996).
Suppose that the central bank may choose a value of $\delta$ for all future devaluations and that its loss function for a devaluation size $\delta$ is

$$L(\delta) = -\int_0^{+\infty} e^{-\beta t} \delta \dot{R}_t \, dt + \sum_{i=1}^{\infty} e^{-\beta t_i} e^{\omega \delta},$$

(19)

where $t_i \equiv$ ith devaluation time.

Here, $\beta > 0$ is a constant discount factor and $\omega > 0$ is also a constant.

The two components of the loss function have an obvious interpretation. The first measures the central bank’s aversion to reserve loss, while the second reflects its reputational cost from devaluing. Closed form solutions for the dynamic system are only available for the steady state, order 1 cycle so we restrict our analysis of the optimal devaluation size to this case.

Assuming that a devaluation of size $\delta$ occurred just before time 0, we know that subsequent devaluations take place at $t = i\delta/\mu$ for integer values, $i$.\textsuperscript{11} We prove in the appendix that:

$$L(\delta) = \gamma \left[ \frac{K_0}{\beta} + \frac{K_1 \left( e^{(\eta_1-\beta)\frac{\delta}{\mu}} - 1 \right)}{(\eta_1 - \beta)(1 - e^{\beta \frac{\delta}{\mu}})} + \frac{K_2 \left( e^{(\eta_2-\beta)\frac{\delta}{\mu}} - 1 \right)}{(\eta_2 - \beta)(1 - e^{\beta \frac{\delta}{\mu}})} \right] + e^{\omega \delta} \frac{e^{\beta \frac{\delta}{\mu}}}{1 - e^{\beta \frac{\delta}{\mu}}},$$

(20)

where $K_0$, $K_1$ and $K_2$ are coefficients given in the appendix.

To analyse how various macroeconomic factors might influence the monetary authorities’ decisions, we minimize the loss function $L(\delta)$ numerically for different choices of the parameters of the model.\textsuperscript{12} Table 5 presents elasticities of the loss function and the optimal devaluation size with respect to parameters of the model. Elasticities are calculated using deviations from a set of benchmark parameter values presented in the second column of the table. As indicated, the optimal devaluation size equals 0.211 (25% in percentage change rather than log differences). Such a value

\textsuperscript{11}Note, we assume that the initial level of the official exchange rate is selected so as to delay as much as possible the first devaluation. This reasonable assumption implies that initially reserves accumulate.

\textsuperscript{12}Note that we again normalize by setting $R = 0$. 

is broadly consistent with devaluations in official exchange rates observed in Latin American countries.

Table 5 shows that two parameters which significantly affect the optimal level of devaluation are the growth rate of domestic credit, $\mu$, and the central bank’s discount rate, $\beta$. The two sets of elasticities are of opposite sign and have almost identical absolute values. From the analytical expression for $L(\delta)$, one may see that $\mu$ and $\beta$ operate in opposite directions in that various terms are of the form: $\exp[(\beta/\mu)\delta)]$. Though similar, the size of the elasticities are not identical, since, in varying $\mu$, we also change the regular cycle dynamics and hence the various $K'_{ij}s$ in the expression for $L_1(\delta)$ (see the Appendix).

The interpretation of elasticities with respect to $\beta$ is clear. Increasing the discount factor leads the central bank to weight future devaluations less. For $\mu$, the opposite argument holds. If the rate of domestic credit growth increases, the discounted cost of future devaluations is larger, since they will occur sooner. This finding is consistent with the observation of Flood and Marion (1994) that in countries with higher inflation rates, the size of realignment of the official rate is generally smaller. Finally, the large elasticity with respect to $\omega$ reflects the obvious fact that increasing the cost of a devaluation reduces the incentive to delay it.

6. Conclusion

We show in this paper how cycles in reserves and exchange rate premia may result from leakages between official and parallel foreign exchange markets. We suggest that failed devaluations may be viewed as part of a longer term process of reserve gain and loss, effectively endogenising what others have regarded as regime shifts.

We provide empirical justification for our approach by presenting the coherences (or frequency domain correlation measures) between reserves, money and exchange rate premia for twelve different Latin American countries in the 1970s and 1980s. These suggest that, in the latter period, Latin American countries experienced reserve
and exchange rate cycles of the kind we describe in our formal modelling.

The models we develop fit the stylized facts listed in Section 2.4. By assumption, reserves fall when the exchange rate premium is large (consistent with stylized fact 1). The exchange premium is increasing in the real money supply (see stylized fact 2). As the system cycles, real money supply increases between devaluations while reserves first increase and then fall. Hence, there is no clear correlation between reserves and money supply (see stylized fact 3). The model predicts that reserves, exchange rate premia and real money supply will tend to cycle together (consistent with stylized fact 4). Under Assumption 1, in the steady state, the economy may follow a single or a pair of regular cycles (consistent with stylized fact 5).

An interesting feature of the cycles we examine is that they may exhibit path dependency, i.e., quite different long run behavior may result even if the initial conditions are close together. In effect, our model shows how a small ‘policy non-linearity’, in this case fully anticipated devaluations in the official exchange rate, can generate asymptotic behavior totally unlike that implied by standard log-linear monetary models.

In principle, one might expect governments in our framework to select the amplitude and frequency of cycles based on the timetable for future elections. Given the level of reserves (which is largely predetermined as far as a government is concerned), the authorities can select a particular future path of reserve and devaluation cycles by selecting an appropriate current official exchange rate.\textsuperscript{13} In the last section of the paper, we examine the dependence of the authorities’ optimal choice of devaluation size on such parameters as its discount rate and the growth rate of domestic credit.

\textsuperscript{13}Interestingly, newspaper accounts have suggested that the current Brazilian government designed its stabilization package prior to the last election so that pressure to devalue could be resisted until after the poll.
Appendix

Spectral and coherence estimation:

To estimate the power spectrum and the coherence, we use the following estimators:

\[
\hat{s}_x(\omega_j) = \frac{\hat{C}_x^2}{2\pi} + \frac{1}{\pi} \sum_{\tau = 1}^{n-1} k_n(\tau) \hat{C}_x^x \cos(\tau \omega_j), \quad (21)
\]

\[
\hat{s}_{xy}(\omega_j) = \frac{1}{2\pi} \sum_{\tau = -n+1}^{n-1} k_n(\tau) \hat{C}_{xy}^x e^{-i\tau \omega_j}, \quad (22)
\]

where, \(n\) is the number of observations, \(\omega_j \equiv j\pi/m\), and where

\[
\hat{C}_x^x \equiv \frac{1}{n - \tau} \sum_{t=1}^{n-\tau} (x_t - \bar{x})(x_{t+\tau} - \bar{x}), \quad (23)
\]

\[
\hat{C}_{xy}^x \equiv \frac{1}{n - \tau} \sum_{t=1}^{n-\tau} (x_t - \bar{x})(y_{t+\tau} - \bar{y}), \quad (24)
\]

\[
k_n(\tau) = \begin{cases} 
1 - \frac{\tau}{m} & \text{for } \tau < m \\
0 & \text{for } \tau \geq m \end{cases}, \quad \text{where } m \approx n/3. \quad (25)
\]

The weighting scheme summarized in equation (25) corresponds to a Bartlett Window. We experimented by also using a Parzen window (see Granger and Newbold (1986)) but this made little difference to the results.

Proof of Proposition 1:

Standard methods yield this solution. See, for example, Hirsch and Smale (1974), Chapter 3. \(\square\)

Proof of Lemma 1:

From Proposition 1, we know that \(\rho_t = \rho_t^* + \tilde{\rho}_t\). Therefore:

\[
\frac{\partial \rho_t}{\partial t} = \frac{\partial \rho_t^*}{\partial t} + \frac{\partial \tilde{\rho}_t}{\partial t}, \quad (26)
\]

where:

\[
\frac{\partial \tilde{\rho}_t}{\partial t} = A_1 \eta_1 \xi_{11} \exp(\eta_1 t) + A_2 \eta_2 \xi_{12} \exp(\eta_2 t), \quad (27)
\]

\[
\frac{\partial \rho_t^*}{\partial t} = -\left( \frac{C_1 \xi_{11}}{\eta_1} + \frac{C_2 \xi_{12}}{\eta_2} \right) \mu + C_1 \xi_{11} \left( Z_0 + \frac{\mu}{\eta_1} \right) \exp(\eta_1 t) + C_2 \xi_{12} \left( Z_0 + \frac{\mu}{\eta_2} \right) \exp(\eta_2 t). \quad (28)
\]
Thus, it follows that:
\[
\frac{\partial \rho_t}{\partial t} = - \left( \frac{C_1 \xi_{11}}{\eta_1} + \frac{C_2 \xi_{12}}{\eta_2} \right) \mu + \xi_{11} \left[ A_1 \eta_1 + C_1 \left( Z_0 + \frac{\mu}{\eta_1} \right) \right] \exp(\eta_1 t) + \xi_{12} \left[ A_2 \eta_2 + C_2 \left( Z_0 + \frac{\mu}{\eta_2} \right) \right] \exp(\eta_2 t).
\]

From (14) and (15), we have that:
\[
C_1 \xi_{11} + C_2 \xi_{12} = 0,
\]
and therefore (29) reduces to:
\[
\frac{\partial \rho_t}{\partial t} = \xi_{11} \left[ A_1 \eta_1 + C_1 \left( Z_0 + \frac{\mu}{\eta_1} \right) \right] \exp(\eta_1 t) + \xi_{12} \left[ A_2 \eta_2 + C_2 \left( Z_0 + \frac{\mu}{\eta_2} \right) \right] \exp(\eta_2 t) \] (30)

We know that \( \eta_1 > 0 \), while \( \eta_2 < 0 \). This implies that in (30) the first term will dominate. Moreover, if \( \xi_{11} > 0 \), then \( C_1 < 0 \), and thus \( \lim_{t \to +\infty} \frac{\partial \rho_t}{\partial t} = +\infty \) if (16) holds. As \( \partial R_t/\partial t = -\gamma \rho_t \) the proof is completed. \( \Box \)

**Proof of Proposition 3:**

From the solution of the dynamic system, we know that:
\[
\rho_t = \tilde{\rho}_t + \rho^*_t \quad \text{where} \quad \rho^*_t = C_1 \int_0^t e^{\eta_1(t-\tau)} \xi_{11} Z_{\tau} \, d\tau + C_2 \int_0^t e^{\eta_2(t-\tau)} \xi_{12} Z_{\tau} \, d\tau.
\]

and \( Z_t = Z_0 + \mu t \). It is not difficult to check that:
\[
C_1 \int_0^t e^{\eta_1(t-\tau)} \xi_{ij} Z_{\tau} \, d\tau = C_i \xi_{ij} e^{\eta_i t} \int_0^t (Z_0 + \mu \tau) e^{\eta_i \tau} \, d\tau = \frac{C_i \xi_{ij}}{\eta_i} \left[ \left( Z_0 + \frac{\mu}{\eta_i} \right) (e^{\eta_i t} - 1) - \mu t \right].
\]

Calculating the eigenvalues and eigenvectors of \( \Phi_1 \) one can easily check that: \( (C_1 \xi_{11}/\eta_1) + (C_2 \xi_{12}/\eta_2) = 0 \). Thus, we have that:
\[
\rho^*_t = \frac{C_1 \xi_{11}}{\eta_1} h(\eta_1, t) + C_2 \xi_{12} \frac{C_2 \xi_{12}}{\eta_2} h(\eta_2, t) \quad \text{where} \quad h(\eta_i, t) = \left( Z_0 + \frac{\mu}{\eta_i} \right) e^{\eta_i t} - \frac{\mu}{\eta_i}.
\]

Hence, adding the expressions for \( \tilde{\rho}_t \) and \( \rho^*_t \) and reordering terms, we obtain:
\[
\rho_t = K_0 + K_1 e^{\eta_1 t} + K_2 e^{\eta_2 t},
\]
where
\[
K_0 = -\mu \left[ \frac{C_1 \xi_{11}}{\eta_1} + \frac{C_2 \xi_{12}}{\eta_2} \right],
\]
\[
K_1 = A_1 + \frac{C_1}{\eta_1} \left( Z_0 + \frac{\mu}{\eta_1} \right) \xi_{11},
\]
\[
K_2 = A_2 + \frac{C_2}{\eta_2} \left( Z_0 + \frac{\mu}{\eta_2} \right) \xi_{12},
\]

(33)

(34)

(35)

(36)
where \( \eta_j \) and \( \epsilon_j \) are respectively the \( j \)th eigenvalue and eigenvector of the matrix \( \Phi_1 \). \( A'_j \)s and \( C'_j \)s are as defined in Proposition 1.

Turning now to reserves, we know that:

\[
R_t = \tilde{R}_t + R_t^* \quad \text{where} \quad R_t^* = C_1 \int_0^t e^{\eta_1(t-\tau)} \xi_{21} Z_\tau \, d\tau + C_2 \int_0^t e^{\eta_2(t-\tau)} \xi_{22} Z_\tau \, d\tau. \tag{37}
\]

Therefore, with calculations similar to those performed above, we can show that:

\[
R_t = J_0 + J_1 e^{\eta_{1t}} + J_2 e^{\eta_{2t}} + J_3 t,
\]

where

\[
J_0 = -\left[ \frac{C_1 \xi_{21}}{\eta_1} \left( Z_0 + \frac{\mu}{\eta_1} \right) + \frac{C_2 \xi_{22}}{\eta_2} \left( Z_0 + \frac{\mu}{\eta_2} \right) \right],
\]

\[
J_1 = \left[ A_1 + \frac{C_1}{\eta_1} \left( Z_0 + \frac{\mu}{\eta_1} \right) \right] \xi_{21},
\]

\[
J_2 = \left[ A_2 + \frac{C_2}{\eta_2} \left( Z_0 + \frac{\mu}{\eta_2} \right) \right] \xi_{22},
\]

\[
J_3 = -\mu \left( \frac{C_1 \xi_{21}}{\eta_1} + \frac{C_2 \xi_{22}}{\eta_2} \right).
\]

Define \( \xi' = [\xi'_{t,j}] \equiv \imath(\xi) \), where \( \xi \) is the matrix whose columns are the eigenvectors of \( \Phi_1 \). Since \( R \) is normalized to zero, \( A_1 = \xi_{11} \rho_0 \) and \( A_2 = \xi_{21} \rho_0 \). Imposing the conditions \( \rho_T = \rho_0 + \delta \) and \( R_T = R_0 = 0 \) yields

\[
\rho_0 + \delta = -\mu \left[ \frac{C_1 \xi_{11}}{\eta_1^2} + \frac{C_2 \xi_{22}}{\eta_2^2} \right] + \xi'_{11} \rho_0 \xi_{11} e^{\eta_{1t}} + \xi'_{21} \rho_0 \xi_{12} e^{\eta_{2t}}
\]

\[
+ \left[ \frac{C_1 \xi_{21}}{\eta_1} e^{\eta_{1t}} + \frac{C_2 \xi_{22}}{\eta_2} e^{\eta_{2t}} \right] Z_0 + \mu \left[ \frac{C_1 \xi_{21}}{\eta_1^2} e^{\eta_{1t}} + \frac{C_2 \xi_{22}}{\eta_2^2} e^{\eta_{2t}} \right], \tag{38}
\]

\[
0 = -\mu \left[ \frac{C_1 \xi_{21}}{\eta_1^2} + \frac{C_2 \xi_{22}}{\eta_2^2} \right] + \xi'_{11} \rho_0 \xi_{21} e^{\eta_{1t}} + \xi'_{21} \rho_0 \xi_{22} e^{\eta_{2t}} - \left[ \frac{C_1 \xi_{21}}{\eta_1} + \frac{C_2 \xi_{22}}{\eta_2} \right] \delta
\]

\[
- \left[ \frac{C_1 \xi_{21}}{\eta_1} (1 - e^{\eta_{1t}}) + \frac{C_2 \xi_{22}}{\eta_2} (1 - e^{\eta_{2t}}) \right] Z_0 + \left[ \frac{\mu C_1 \xi_{21}}{\eta_1^2} e^{\eta_{1t}} + \frac{\mu C_2 \xi_{22}}{\eta_2^2} e^{\eta_{2t}} \right]. \tag{39}
\]

Reordering terms gives

\[
\Gamma = \Sigma \begin{bmatrix} \rho_0 \\ Z_0 \end{bmatrix}, \tag{40}
\]

where:

\[
\Gamma = \Sigma \begin{bmatrix} \delta + C_1 \rho_0 \xi_{11} (1 - \exp(\eta_1 T)) + C_1 \rho_0 \xi_{12} (1 - \exp(\eta_2 T)) \\ C_1 \rho_0 \mu T + C_1 \rho_0 \xi_{21} (1 - \exp(\eta_1 T)) + C_2 \rho_0 \mu T + C_2 \rho_0 \xi_{22} (1 - \exp(\eta_2 T)) \end{bmatrix}. \tag{41}
\]

23
and \( \Sigma \) is defined as in equation (18). The solution of (40) is the equilibrium values for \( \rho_0 \) and \( Z_0 \). \( \Box \)

Proof of Proposition 4:

Suppose \( N > 2 \). Denote the terminal points of the \( N \) orbits corresponding to the individual cycles: \( b_1, b_2, \ldots, b_N \), where \( i < j \) for \( b_i < b_j \). The order-preserving property of Poincaré maps and Assumption 1 imply that the cycle ending at \( b_1 \) must be followed by a cycle ending at \( b_N \). As \( b_N \) is the maximum level prior to a devaluation and \( b_1 \) is the minimum, the cycle ending at \( b_N \) must be followed by one ending at \( b_1 \). Hence, \( N = 2 \), a contradiction.

To prove the second part of the proposition, we begin by showing that, for any initial condition, \( Z_0 \), when \( R_0 = R \), the system must converge either to a regular cycle of order 1 or 2. Again, let \( b_1, b_2, \ldots, b_N \), where \( i < j \) for \( b_i < b_j \) denote the terminal points of the \( N \) orbits corresponding to successive individual cycles. Consider the case that \( b_1 \) is smaller than both \( b_2 \) and \( b_3 \). We have two possibilities: (i) \( b_1 < b_3 < b_2 \), and (ii) \( b_1 < b_3 = b_2 \). Note that \( b_3 \) cannot be larger than \( b_2 \) otherwise the orbits intersect and the uniqueness of the solution of the two point boundary problem is violated.

In (ii), we have an order-one cycle and the statement is proved. From \( b_1 < b_3 < b_2 \) it follows that \( b_1 - \delta < b_3 - \delta < b_2 - \delta \). As the orbits cannot cross, this, in turn, implies that: \( b_3 < b_4 < b_2 \); with a similar argument, we conclude that \( b_5 < b_6 < b_4 \). Reordering, we have:

\[
\begin{align*}
  b_1 < b_3 < b_5 < b_6 < b_4 < b_2.
\end{align*}
\]

In general, given an index \( i \) we have; if \( i \) is odd: \( b_i < b_{i+2} < b_{i+1} \). If \( i \) is even \( b_{i+1} < b_{i+2} < b_i \). This couple of inequalities implies that we have the following sequence of inequalities; assuming that \( i \) is odd:

\[
\begin{align*}
  b_i < b_{i+2} < b_{i+4} < \ldots < b_{i+2n} < \ldots < b_{i+2n-1} < \ldots < b_{i+5} < b_{i+3} < b_{i+1}.
\end{align*}
\]

Therefore, we can introduce the sequence \( \{ |b_{i+1} - b_i| \}_{i=1}^{+\infty} \) that is monotonically decreasing with a lower bound in zero. This implies that \( \{ |b_{i+1} - b_i| \}_{i=1}^{+\infty} \) converges to a limit \( k \). Given that, for \( i \) odd, \( b_i < b_{i+2} < b_{i+1} \), the only possible value of \( k \) is zero. Hence, we conclude that the sequence of values \( \{ b_i \}_{i=1}^{+\infty} \) converges. A similar argument holds if \( b_2 \) and \( b_3 \) are smaller than \( b_1 \).

We now suppose that the inter-devaluation times are fluctuating. There are two cases: (i) \( b_2 < b_1 < b_3 \), and (ii) \( b_3 < b_1 < b_2 \). Let us concentrate on (i). For any index \( i \geq 3 \) such
that $b_i > b_1$, we have that $b_i - \delta \in (b_1 - \delta, Z^*)$. Hence, the cycle ending at $b_{i+1}$ must be everywhere to the left of the cycle that ended at $b_1$, and therefore $b_{i+1} \in [Z^*, b_2)$. It follows that $b_{i+1} - \delta < b_2 - \delta$, and hence the cycle ending at $b_{i+2}$ must be everywhere to the right of the one which ends at $b_1$ and so $b_{i+2} > b_1$.

Since the relation $b_i > b_1$ implies that $b_{i+1} < b_1 < b_{i+2}$, $b_i$, $\forall i$ and that $b_3 > b_1$, we have that for $i$ odd $b_i > b_1$ and for $i$ even $b_i < b_1$. Thus, successive cycles are in disjoint regions of state space. A similar argument holds in case (ii). We now use this fact to show that under fluctuating devaluation times, the system must converge to an order 2 cycle.

For any index $i$, we define $b_{i\text{max}} \equiv \max_{j \leq i} \{b_j\}$ and $b_{i\text{min}} \equiv \min_{j \leq i} \{b_j\}$. Suppose $b_i = b_{i\text{max}}$. Hence, $b_i - \delta > b_j - \delta$ for all $j < i$. This implies that $b_{i+1} < b_j \forall j < i$ and that $b_{i+2} > b_{i\text{max}}$. Now, in case (i) of the last paragraph, $b_3 = b_{3\text{max}}$, so long as the system is not in a regular cycle, $\{b_{i\text{max}}\}_{i=1}^{+\infty}$ must be a strictly monotonically increasing sequence. By assumption, this sequence is bounded by $Z^* + \delta$, and, hence, the sequence converges to $k_{\text{max}} \in (b_1, Z^* + \delta]$. A similar argument shows that $\{b_{i\text{min}}\}$ is monotonically decreasing and bounded by $Z^*$, and therefore converges to $k_{\text{min}} \in [Z^*, b_1)$.

We have therefore shown that, for any initial condition, the system must converge to an order 1 or an order 2 regular cycle. Now, if there exists a regular cycle of order 2, there must also clearly be a regular cycle of order 1. Consider the levels of domestic credit in foreign prices which pertain just after the devaluation in the regular 1 cycle, $Z^*_1$, and after the shorter of the two cycles in the order 2 cycle, $Z^*_2$. Clearly, $Z^*_2 < Z^*_1$. For initial conditions in $[Z^*_2, Z^*_1]$, we have seen that the system must converge either to an order 1 or to an order 2 cycle. Clearly, for any $\epsilon$ we will be able to find $Z_{1\epsilon}$ and $Z_{2\epsilon}$ such that $|Z_{1\epsilon} - Z_{2\epsilon}| < \epsilon$ and starting at $Z_{1\epsilon}$, $R$ and $Z_{2\epsilon}$, $R$ the system converges to order 1 and order 2 regular cycles respectively. □

**Derivation of the loss function:**

One may show that:

$$
\mathcal{L}_1(\delta) \equiv \int_0^\infty e^{-\beta t} \rho_t \, dt = \frac{\int_0^\delta e^{-\beta t} \rho_t \, dt}{1 - e^{-\beta \mu}}, \quad (42)
$$

$$
\mathcal{L}_2(\delta) \equiv \int_0^\infty e^{-\beta t} I_t \, dt = \frac{e^{\beta \mu}}{1 - e^{-\beta \mu}}, \quad (43)
$$

25
From the proof of Proposition 3, we can deduce the path of $\rho_t$ over the steady state cycle. Substituting $\rho_t$ in $L_1$ and solving the integral, we obtain:

$$L_1(\delta) = \frac{K_0}{\beta} + \frac{K_1}{(\eta_1 - \beta)(1 - e^{\beta \delta})} \left( e^{(\eta_1 - \beta) \frac{\delta}{\tau}} - 1 \right) + \frac{K_2}{(\eta_2 - \beta)(1 - e^{\beta \delta})} \left( e^{(\eta_2 - \beta) \frac{\delta}{\tau}} - 1 \right).$$

Notice that the constants $K_j$'s depend on the level of domestic credit in foreign prices at the start of the cycle, $Z_0$. This was derived in Proposition 3. □
References


De Vries, M. 1965, Multiple exchange rates: expectations and experiences, IMF Staff Papers, 12, 282-313.


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Note: $\phi_{RP}$ denotes the correlation coefficient between changes in log reserves and the level of the log exchange rate premium.
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Note: $\phi_{PM}$ denotes the correlation coefficient between changes in the log exchange rate premium and changes in log money.
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Note: $\phi_{RM}$ denotes the correlation coefficient between changes in log reserves and changes in log money.
Table 4: Steady state order 1 cycle comparatic statics

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<th>Symbol</th>
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<td>0.00</td>
<td>0.01</td>
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<td>Reserve Minimum</td>
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Note: entries are elasticities of the min. and max. over the cycle of the exchange rate premium ($\rho$) log reserves $R$ and log domestic credit $Z$ with respect to changes in the relevant parameter.
Table 5: Optimal devaluation size comparative statics

<table>
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<tr>
<th>Parameter</th>
<th>Symbol</th>
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<th>$\delta^*$</th>
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Note: entries are the elasticities of the loss function, $\mathcal{L}(\delta)^*$, and of the optimal devaluation, $\delta^*$, with respect to the model parameters. For the benchmark values of the parameters, $\mathcal{L}(\delta)^*$ and $\delta^*$ are equal to 25.809 and 0.211 respectively.